The acceleration due to gravity

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We conduct a free-fall experiment to measure the acceleration due to gravity g. We measure a value of $g = 9.865 \pm 0.111$ m/s^2, which agrees quite well ($+0.595\,\%$) with the international standard definition: $g = 9.80665$ m/s^2 [1]. This result required careful analysis of systematic errors in the data, to account for the small, unintentional downward angle of the projectile launcher.

NOTE TO STUDENTS

This lab report is intended to be a standalone document. As such, it includes many features which I am not expecting from students, like: an abstract, a preface, an overly thorough introduction, sophisticated error analysis, an unnecessary appendix, and beautiful typesetting via LaTeX. Nonetheless, this document is a good example of what an *ideal* lab report looks like.

PREFACE

When children learn to walk, they will fall many times. Luckily they are not very tall, so when they fall it is not far, and there is no time to accelerate to high speed. Young children also have the advantage of being unmassive. When the ground arrests their fall, their paltry inertia gives rise to a small impact force. Unharmed, they pick themselves up and keep toddling.

But when adults fall, they fall hard. Ten years ago the author slipped on mud while running. To catch himself he straight-armed the ground. As the force was transmitted at the elbow, his arm bones slammed together forcefully enough to chip the tip of his radius bone. Surgery was required, and range-of-motion was lost. His lesson? Fall like a sack of potatoes.

This anecdote reminds us that gravity — like the sea and the sun — is relentless and uncaring; it will smash us at the first opportunity. But like the sea and the sun, gravity is also the giver of life; it caused the earth to form and keeps the atmosphere, and us, from floating off into space. As both a creator and a destroyer, gravity demands our respect. This is why we study it.

I. INTRODUCTION

One of gravity's defining properties is somewhat counter-intuitive; gravity, which acts proportional to mass, accelerates a mouse and a man at the same rate. This result is obtained by using Newton's law of universal gravitation to calculate the gravitational force F acting on an object of mass m at the surface of the earth (using Newton's gravitational constant G and the mass M and radius R of the earth)[2]. We then use Newton's 2$^{\text{nd}}$ law to convert this force into an acceleration a

$$F = \overbrace{\underset{\substack{\uparrow \\ \text{inertial mass}}}{m}\,a}^{2^{\text{nd}}\text{ law}} = \overbrace{\underset{\substack{\uparrow \\ \text{gravitational mass}}}{m}\left(\frac{G\,M}{R^2}\right)}^{\text{universal gravitation}} = m\,g\,. \tag{1}$$

Note that both the gravitational force exerted on the object, and the object's inertia (resistance to motion), are proportional to its mass. Canceling out Eq. 1's common m (and bundling its constants into g, the standard acceleration due to gravity), we find that

$$a = g\,. \tag{2}$$

The earth's gravity accelerates all objects equally.[1]

To measure g, we can observe a test object falling under the influence of gravity. The apparatus at hand is a spring-loaded launcher which fires a steel ball bearing at 3–6 m/s. We define the y-axis to be orthogonal to the floor and the x-axis to be along the floor, directly underneath the ball's flight path. Since g is effectively constant,[2] and choosing to ignore air resistance, we can use the traditional equations of motion [2]. Our choice of coordinate system gives $a_x = 0$ and $a_y = -g$, and we are free to place the origin at the launch point so that

$$x = v_{x0}\,t\,, \tag{4}$$

$$y = v_{y0}\,t - \frac{g}{2}\,t^2\,. \tag{5}$$

[1] In a normal setting, a feather and a bowling ball experience different air resistance as they fall, and therefore accelerate at different rates. In a vacuum chamber they fall in unison [3].

[2] There is no time dependence in g, and its altitude dependence is negligible on the scale of our experiment. We can prove this by defining $y = 0$ at the earth's radius and using a first-order series expansion to find the value of g at different altitudes.

$$g(y) = \frac{GM}{(R+y)^2} \approx g(1 - 2y/R)\,. \tag{3}$$

This approximation is accurate provided that $(2y/R) \lesssim 10^{-2}$. The ball will fall about 1 meter, so $(2\Delta y/R) \approx 3 \times 10^{-7}$. Thus, we expect g to vary by about 0.3 ppm during the experiment, an effect several orders of magnitude smaller than air resistance, and thus completely negligible.

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The launcher's elevation angle θ relative to the floor is adjustable. Projecting the initial velocity into the x and y directions, we obtain

$$v_{x0} = v_0 \cos\theta \,, \tag{6}$$
$$v_{y0} = v_0 \sin\theta \,. \tag{7}$$

To conduct a free-fall experiment, we orient the launcher at zero elevation ($\theta = 0$) so that $v_{y0} = 0$ and $v_{x0} = v_0$. Since the physics of the two axes are completely independent, the ball moves in the y-direction as if it were simply dropped, allowing us to extract

$$g = -\frac{2y}{t^2} \,. \tag{8}$$

The advantage of firing the ball from the launcher, instead of simply dropping it, is that we do not have to construct a dropping apparatus that instantaneously releases the ball (and does so without any residual interactions in the first few moments of the drop); the launcher is already such a device.

The downside of this experimental setup is its reliance on very accurately setting $\theta = 0$. Any error while setting up the launcher — either via a miscalibrated or imprecise measurement of θ — will manifest as a non-zero v_{y0}, which alters the time-of-flight (TOF) t. Naïvely using Eq. 8 on such data will cause v_{y0} to infect the extracted g. For example, if $\theta > 0$, then it will take longer for the ball to reach the ground than if the ball were dropped from rest. This will be interpreted by Eq. 8 as a weaker g. And since this effect scales with v_0, the extracted g will become even weaker as v_0 increases.

This v_0 dependence is actually a blessing; we can use it to determine the actual θ of the launcher and correct the g measurement. This is accomplished by rearranging Eq. 5 into a linear model to fit the data

$$\left(\frac{2y}{t^2}\right) = \underset{\underset{m}{\uparrow}}{\sin\theta} \left(\frac{2v_0}{t}\right) - \underset{\underset{b}{\uparrow}}{g} \,. \tag{9}$$

The slope of the fit ($m = \sin\theta$) will tell us the unknown launch angle and the intercept ($b = -g$) will give a much more accurate measurement of g. Since we do not actually measure v_0 (only $v_{x0} = x/t$), we should insert $v_0 = v_{x0}/\cos(\theta)$ into Eq. 9 to give the final fit equation

$$\left(\frac{2y}{t^2}\right) = \tan\theta \left(\frac{2x}{t^2}\right) - g \,. \tag{10}$$

II. EXPERIMENTAL METHODS

The spring launcher was securely fastened to the laboratory table and adjusted to zero elevation ($\theta = 0$) via the launcher's small, integral plumb bob and protractor. To minimize parallax error when reading the angle from the protractor, the author's line of sight was kept orthogonal to the scale. Nonetheless, due to the small size of the protractor, and the tiny weight of the plumb bob, its precision was estimated to be $\pm 1°$.

The launch point (marked with crosshairs near the muzzle of the launcher) was used as the origin of the coordinate system. A 1 lb plumb bob was used to locate the x-origin — the point on the floor directly below the launch point — which was marked with a pencil. A measuring tape (mm precision) was used to record the height of the launch point above the floor, which was translated to the floor's position: $y = -107.25 \pm 0.25$ cm.

The time-of-flight measurement apparatus consisted of: (i) a photo-gate broken by the ball immediately after launch, to start the clock and (ii) a sensor pad on the floor which stopped the clock when struck by the ball. The height of the sensor pad was measured to be $h = 1.75 \pm 0.1$ cm. The TOF system was attached to the computer and automatically calibrated by the Pasco Captsone DAQ software.

To measure the x-distance traveled by each ball, a letter-sized piece of white printer paper was taped to the front of the sensor pad. Before each launch, a piece of carbon paper was placed on top of the white paper, to create a small black mark when struck by the ball. After each trial, the fresh mark was sequentially numbered. The total distance $x = x_{\text{large}} + x_{\text{small}}$ was composed of x_{large}, the distance from the x-origin to the edge of the paper, and x_{small}, the distance from the edge of the paper to each black mark.

The launcher had three v_0 settings, delineated by the number of clicks heard (1, 2 or 3) while cocking the plunger with the cocking rod (the click was caused by the launch bar snapping into a restraining notch). The ball was launched by using a lanyard to swiftly raise the launch bar.

Before taking data for each v_0 setting, a dry run was conducted to find the approximate location where the balls would land. Then the sensor was positioned so that the white paper was aligned with the edge of one of the floor tiles. This allowed the sensor to be reset after each launch, since the ball's impact occasionally knocked it out of position.

Four trials were taken per v_0 setting, and each TOF was recorded. Before moving to the next v_0, the measuring tape was used to record x_{large}. Because there was only one experimenter to hold the tape, the uncertainty of x_{large} was ± 1 cm. After completing all trials, the white paper was removed from the sensor pad and taken to the table to measure x_{small} with a meter stick (mm precision). The edge (zero) of the meter stick was worn, so all x_{small} readings were taken by using the 10 cm mark as the zero. The absolute reading was recorded, with the 10 cm offset subtracted during analysis. However, the imprecision of x_{large} spoils the extra precision of x_{small}.

Each grouping of four trials exhibited a spread in the z-direction ($\hat{z} = \hat{x} \times \hat{y}$) of half-width $\delta z = \sim 1.5$ cm. The relative error from ignoring z when calculating the total flight distance is $\epsilon \approx \frac{x}{\sqrt{x^2 + \delta z^2}} - 1 \approx \frac{-\delta z^2}{2x^2}$ (using a second-order series approximation). The slowest v_0 exhibited

$x \approx 1.5$ m, which gives $\epsilon = \mathcal{O}(-10^{-6})$. This error is several orders of magnitude smaller than the error made by ignoring air resistance, so that z is safe to ignore.

III. RESULTS AND DISCUSSION

Each trial is specified by the ball's TOF t and its impact position (x, y) on the sensor pad (although every trial uses the same $y = -1.055 \pm 0.003$ m). Assuming that $\theta = 0$, the best strategy to mitigate random errors is to average x and t before calculating v_0 and g.

$\bar{x}$ (m)	$\bar{t}$ (s)	v_0 (m/s)	g (m/s^2)
1.460 ± 0.010	$0.4562 \pm 4.0\text{E}-4$	3.200 ± 0.022	10.14 ± 0.03
1.901 ± 0.014	$0.4556 \pm 3.0\text{E}-3$	4.170 ± 0.028	10.16 ± 0.03
2.473 ± 0.013	$0.4523 \pm 1.4\text{E}-3$	5.468 ± 0.026	10.31 ± 0.03

Weighted average g: 10.20 ± 0.02 m/s^2

TABLE I. Results from averaging data, assuming $\theta = 0$. Eq. 6 is used to calculate v_0 and Eq. 5 to calculate g. Errors in $\bar{t}$ are estimated from the statistical error (unbiased variance σ_t^2). The $\bar{x}$ error sums the statistical and systematic errors in quadrature (x_{large} has a measurement error of ± 1 cm). Errors in the results are estimated by propagating errors using variance and covariance [4] (and are dominated by the errors in x and y). The weighted average $\bar{g}$ uses weights $w_i = \sigma_i^{-2}$, with $\sigma_{\bar{g}} = \sqrt{1/\sum_i w_i}$ [4].

Table I depicts the initial results. The measurement of $g = 10.20 \pm 0.02$ m/s^2 is 4.01% larger than the interational standard definition $g = 9.80665$ m/s^2 (which is defined to have its exact value, with no uncertainty)[1]. Although local variations in g are expected (e.g. local altitude, local shape and mass distribution of the Earth), these should be much smaller than 4% (otherwise one could lose 10 lbs by simply moving to a new city). Furthermore, the uncertainty in our measurement of g is *20 times* smaller than its difference from the standard value; the difference is statistically significant.

The initial measurement of g is significantly *high*, which implies that the launcher may have been angled slightly downward. A more complete analysis allowing $\theta \neq 0$ is shown in Fig. 1, which fits Eq. 10's linear model to the data. *Raw* data is fit because x and t are correlated (as they should be, $\mathrm{d}x/\mathrm{d}t = v_{x0}$). A limitation in using raw data is the inability to assign a meaningful uncertainty to the TOF, since variations in arrival time could arise either from imprecision in the TOF apparatus or from variations in v_0. As such, we do not utilize error bars during the fit, giving every data point equal weight.

The best fit extracts $g = 9.865 \pm 0.111$ m/s^2, which is only +0.59% larger than the international standard. In spite of our best efforts to set the launcher to zero elevation, the fit indicates that it was set to $\theta = -0.01805 \pm 0.0058$ (or $-1.034° \pm 0.330°$). Accounting for this error permits the order of magnitude improvement in the accuracy of our g measurement.

Ironically, we pay for 10 times better accuracy with 10 times worse relative precision. This is due to the relatively low R^2 of the fit, which stems primarily from the vertical spread around each v_0 setting. For example, if the main outlier is removed, the fit tightens to $R^2 = 0.77$. However, it was not removed because it did not pass Chauvenet's criteria for removal [4] (since its $n = 0.62 > 0.5$). Note that the apparent intensity of the vertical spread, versus the horizontal spread, is mostly due to the scale of the axes (the vertical axis is zoomed in about 10 times more).

A possible explanation for the vertical spread can be seen by drawing the error band of the fitted launch angle in Fig. 1. Given this visual aid, the vertical spread suggests that individual trials launch at slightly different angles, which could be caused by vibrations during launch perturbing the launch angle (especially at large v_0). In fact, a vibration of 0.33° seems quite reasonable, given that the launcher is neither very heavy nor incredibly secure (even after tightly screwing the half dozen knobs and wingnuts, there was still a moderate amount of play in the test rig).

IV. CONCLUSION

By launching a ball bearing nearly parallel to the ground, we conduct a free-fall experiment and measure an acceleration due to gravity of $g = 9.865 \pm 0.111$ m/s^2, in close agreement (+0.592 %) with the international standard definition of $g = 9.80665$ m/s^2. This result accounts for the dominant systematic error in our experiment; even though we though we set the launcher to zero elevation, it was likely at a slight downward angle of $\theta = -1.034°$. This g measurement does not account for air resistance, the sub-dominant systematic error.

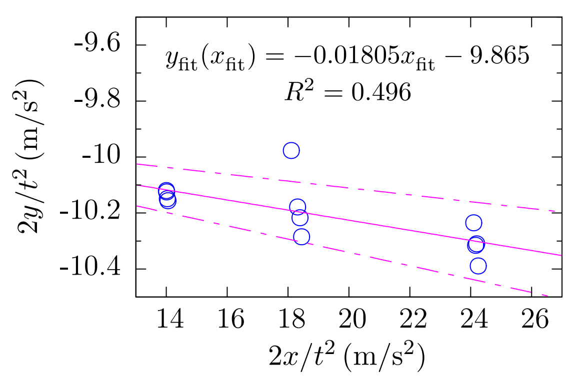

FIG. 1. Raw data fit to Eq. 10 by minimizing the χ^2. Each data point is given equal weight, so that errors in the fit are estimated from the covariance matrix of the final fit (the "asymptotic standard error"). Dot-dashed lines depicting the error band of the slope flank the best fit (solid).

[1] I. O. for Standardization, *ISO 80000-3 : Quantities and Units: Part 3: Space and Time* (ISO, 2006).

[2] H. C. Ohanian and J. T. Markert, *Physics for Engineers and Scientists (Third Edition) (Vol. 1)* (W. W. Norton & Company, 2006).

[3] BBC, "Brian cox visits the world's biggest vacuum chamber," https://youtu.be/E43-CfukEgs?t=2m38s (2014).

[4] J. Taylor, *Published by University Science Books, 648 Broadway, Suite 902, New York, NY 10012, 1997.* (University Science Books, 1997).

Appendix A: The optimal firing angle

NOTE: *My experiment did not test the optimal firing angle. This appendix is for their benefit of my student's.*

Launching from the origin (with a fixed v_0) at some target at height y, what is the *optimal* launch angle θ_{opt} (the one that maximizes our distance from the target)?

We first solve the projectile's equation of motion

$$y = v_0 \sin(\theta) t - \frac{g}{2} t^2 \tag{A1}$$

for the time to reach the target

$$t(\theta) = \frac{v_0 \sin\theta}{g} \left(1 \pm \sqrt{1 - \frac{2gy}{v_0^2 \sin^2\theta}} \right) . \tag{A2}$$

These t solutions are real whenever the parabolic path actually passes through y (i.e. $2gy < v_0^2 \sin^2\theta$). However, just because t is real does not mean t is in the future ($t > 0$), as required by the definition of the problem. Specifically, if $\theta < 0$, only the $\ominus$ solution can be in the future (but doesn't have to be, if $y > 0$). On the other hand, if $\theta > 0$, both solutions (if real) must be in the future. This means that the $\oplus$ time *must* correspond to the farthest distance.

By calculating the distance to the target

$$D(\theta) = v_0 \cos(\theta)\, t(\theta) , \tag{A3}$$

we can find the optimal angle by solving

$$\frac{\mathrm{d}D(\theta)}{\mathrm{d}\theta} = 0 . \tag{A4}$$

However, while this calculation gives the correct answer, it is a nightmare to work out. There is a better way.

Imagine that the target is very close (D is small) and v_0 is more than sufficient to hit it. There are clearly two angles we can use; a high, lobbing shot (like a mortar) or a very low, direct shot (like a rifle). As we move the target further away, there continue to be two angles, but they get closer to each other. When we eventually reach the optimal angle — the farthest distance D that can still hit the target — the two angles converge. Now only one angle can do the job. Hence, if we can solve for the two angles using the quadratic formula, then the optimal angle corresponds to the solution where the root term vanishes, and there is only one solution.

If we assume that we know D, we can solve for the time to hit the target

$$t = \frac{D}{v_0 \cos\theta} . \tag{A5}$$

Plugging this t into the equations of motion (and using $\cos^{-2}\theta = 1 + \tan^2\theta$), we find

$$y = D \tan\theta - \frac{gD^2}{2v_0^2}(1 + \tan^2\theta) . \tag{A6}$$

Now defining $w \equiv \tan\theta$ and $\alpha \equiv (gD^2)/(2v_0^2)$, we obtain a rather simple quadratic equation

$$\alpha\, w^2 - D\, w + (y + \alpha) , \tag{A7}$$

and can immediately find its solutions

$$w = \frac{D}{2\alpha} \left(1 \pm \sqrt{1 - \frac{4\alpha(y+\alpha)}{D^2}} \right) . \tag{A8}$$

The optimal angle occurs when the root term is null, so we can solve for the D which sends the root term to zero. After some algebra we find

$$D = \frac{v_0}{g} \sqrt{v_0^2 - 2gy} . \tag{A9}$$

We have just phrased D in terms of the other degrees of freedom; it is no longer an assumption. Plugging this D into Eq. A6 (and after some more algebra) we obtain

$$\theta_{\mathrm{opt}} = \arctan\left(\frac{v_0}{\sqrt{v_0^2 - 2gy}} \right) . \tag{A10}$$

As a sanity check, we can try $y = 0$; as expected, we obtain $\theta = \pi/4$ (45°).

There is an interesting aspect to this solution. Using conservation of energy [2], we can show that

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v_{\mathrm{impact}}^2 + mgy , \tag{A11}$$

which gives

$$v_{\mathrm{impact}} = \sqrt{v_0^2 - 2gy} . \tag{A12}$$

Thus, we can rephrase the optimal angle in terms of the initial and impact velocities

$$\theta_{\mathrm{opt}} = \arctan\left(\frac{v_0}{v_{\mathrm{impact}}} \right) . \tag{A13}$$

Of course, this result is not very useful for real-world projectiles (like artillery shells) because it leaves out important effects like wind and air resistance. We ignore these in introductory mechanics because simply adding air resistance requires non-linear differential equations.